FREEZING OF A LIQUID AT A FLAT WALL

A. M. Makarov, V. V. Leonov,
V. I. Dubovik, and G. N. Shvedova

UDC 536.421.4

The single-phase Stefan problem is considered with boundary conditions of the second and of the third kind. An iteration scheme is developed for solving the nonlinear integrodifferential equation.

An analysis of the freezing-process dynamics in the case of a warm liquid at a cold solid wall reduces to a problem in which the boundary for the equation of heat conduction is unknown (the Stefan problem). An analytical solution of the Stefan problem in a sufficiently general form involves vast difficulties of a mathematical nature. Several approaches to the solution of this problem are known: a reduction of the Stefan problem to various functional systems of equations [1, 2, 3, 4], a solution in series in "instantaneous" or in "local" eigenfunctions of the problem [5, 6, 9], and the method of integral transformations [7]. None of these methods is very useful for obtaining specific results within a finite interval of the time variable.

The problem of a warm liquid freezing at an isothermal cold wall (boundary condition of the first kind) has been solved in [8] by the method of successive approximations. In this article we will use the iteration method for solving the Stefan problem with boundary conditions of the second and the third kind. Let a liquid filling the half-space $x > 0$ be maintained at a constant temperature T_e higher than the phase transition temperature T_f. Beginning at time $t = 0$, a certain temperature field is maintained on an infinitely large plate at $x = 0$. On plane $x = 0$ there forms ice of density ρ and latent heat of fusion L. The thickness of the frozen layer $x = \delta(t)$ and the temperature distribution $T(x, t)$ are the unknown functions here. With these assumptions, then, we have the single-phase Stefan problem:

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \text{ in } D, \tag{1}$$

$$k \frac{\partial T}{\partial x}\Big|_{x=0} = F\left[t,\ T(0,\ t)\right],\ t > 0, \tag{2}$$

$$T|_{x=\delta(t)} = T_f = \text{const}, \tag{3}$$

$$k \frac{\partial T}{\partial x}\Big|_{x=\delta(t)} = h_e(T_e - T_f) + \rho L \frac{d\delta}{dt},\quad t > 0, \tag{4}$$

$$\delta(0) = 0. \tag{5}$$

Here $F[t, T(0, t)]$ is a certain function of the given arguments and $D = \{x, t;\ 0 < x < \delta(t),\ 0 < t < t_0 < \infty\}$ is the open-ended region where a solution is to be obtained. The second condition (4) at the unknown boundary makes allowance for the effect of convective heat transfer between warm liquid and layer of ice on the freezing rate.

1. Freezing of a Liquid at a Flat Wall from Which a Certain Heat Flux is Extracted

In this case function $F[t, T(0, t)]$ becomes

$$F\left[t,\ T(0,\ t)\right] = q_w = \text{const}, \tag{1.1}$$

where q_w is the heat flux extracted from plane $x = 0$.

N. E. Bauman Technical College, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 21, No. 3, pp. 537-546, September, 1971. Original article submitted December 8, 1970.

TABLE 1. Results of the First, the Second, and the Third Approximation for Function $\tau = \tau(\Delta)$

	α, β	Δ								
		0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
τ_1	$\alpha=0{,}03$	0,1011	0,2026	0,3043	0,4064	0,5088	0,6114	0,7144	0,8176	0,9213
τ_2	$\beta=0{,}01$	0,1011	0,2026	0,3043	0,4064	0,5088	0,6114	0,7144	0,8176	0,9211
τ_3		0,1011	0,2026	0,3043	0,4064	0,5087	0,6114	0,7143	0,8176	0,9211
τ_1	$\alpha=0{,}6$	0,1040	0,2141	0,3303	0,4525	0,5808	0,7151	0,8555	1,0020	1,1545
τ_2	$\beta=0{,}01$	0,1039	0,2132	0,3275	0,4465	0,5692	0,6961	0,8265	0,9605	1,0977
τ_3		0,1019	0,2103	0,3251	0,4459	0,5726	0,7084	0,8422	0,9845	1,1312
τ_1	$\alpha=0{,}6$	0,2503	0,5015	0,7533	1,0060	1,2593	1,5135	1,7683	2,0240	2,2803
τ_2	$\beta=0{,}6$	0,2503	0,5015	0,7533	1,0059	1,2593	1,5134	1,7682	2,0238	2,2801
τ_3		0,2503	0,5014	0,7532	1,0058	1,2592	1,5132	1,7681	2,0236	2,2800

The solution to problem (1)-(5) with condition (1.1) will now be

$$k\frac{\partial T}{\partial x} = k\frac{\partial T}{\partial x}\bigg|_{x=0} + \rho c\int\limits_0^x \frac{\partial T}{\partial t}dx.$$

With conditions (2) and (1.1) we have

$$k\frac{\partial T}{\partial x} = q_w + \rho c\int\limits_0^x \frac{\partial T}{\partial t}dx. \tag{1.2}$$

Writing Eq. (1.2) for $x = \delta(t)$ and taking into account condition (4), we will obtain after a few simple transformations:

$$\frac{d\delta}{dt} = \frac{q_w - h_e(T_e - T_f)}{\rho L} + \frac{c}{L}\int\limits_0^\delta \frac{\partial T}{\partial t}dx. \tag{1.3}$$

We then integrate Eq. (1.2) over the variable x from x to $\delta(t)$ and, together with condition (3), this yields

$$T = T_f - \frac{q_w}{k}(\delta - x) - \frac{\rho c}{k}\int\limits_x^{\delta(t)}\int\limits_0^x \frac{\partial T}{\partial t}dxdx. \tag{1.4}$$

We note that $\partial/\partial t = \partial/\partial\delta \cdot d\delta/dt$, where $d\delta/dt$ is independent of the space variable.

In this case Eq. (1.3) yields

$$\frac{d\delta}{dt} = \frac{q_w - h_e(T_e - T_f)}{\rho L\left(1 - \frac{c}{L}\int\limits_0^\delta \frac{\partial T}{\partial \delta}dx\right)}. \tag{1.5}$$

Taking into account (1.5), we can write Eq. (1.4) as

$$T = T_f - \frac{q_w}{k}(\delta - x) - \frac{c\left[q_w - h_e(T_e - T_f)\right]}{kL\left(1 - \frac{c}{L}\int\limits_0^\delta \frac{\partial T}{\partial \delta}dx\right)}\int\limits^\delta\int\limits^x \frac{\partial T}{\partial \delta}dxdx. \tag{1.6}$$

The right-hand side of Eq. (1.5) is a function of δ and does not explicitly depend on t, which, together with condition (5), allows us to rewrite the expression for $t = t(\delta)$ as

$$t = \frac{\rho L}{q_w - h_e(T_e - T_f)}\left[\delta - \frac{c}{L}\int\limits_0^\delta\int\limits_\delta^\delta \frac{\partial T}{\partial \delta}dxdx\right]. \tag{1.7}$$

A solution to the integrodifferential equation (1.6) can be obtained by the method of successive approximations, whereupon expression (1.7) will yield the time during which an ice layer of thickness δ freezes at the infinitely large plate $x = 0$.

Further calculations will be more conveniently performed with dimensionless quantities. Let

$$\theta = \frac{T - T_f}{T_e - T_f}; \quad z = \frac{k(T_e - T_f)}{q_w}; \quad \Delta = \frac{\delta}{x}; \quad \xi = \frac{x}{z};$$

$$\tau = \frac{q_w}{z\rho L}; \quad \alpha = \frac{c(T_e - T_f)}{L}; \quad \beta = \frac{h_e(T_e - T_f)}{q_w}.$$

Equations (1.6) and (1.7) become now (1.8) and (1.9) respectively:

$$\theta = \xi - \Delta + \frac{\alpha(1-\beta)\int\limits_{\xi}^{\Delta}\int\limits_{0}^{\xi}\frac{\partial\theta}{\partial\Delta}\,d\xi d\xi}{1 - \alpha\int\limits_{0}^{\Delta}\frac{\partial\theta}{\partial\Delta}\,d\xi}, \tag{1.8}$$

$$\tau = \frac{1}{1-\beta}\left(\Delta - \alpha\int\limits_{0}^{\Delta}\int\limits_{0}^{\Delta}\frac{\partial\theta}{\partial\Delta}\,d\xi d\Delta\right). \tag{1.9}$$

As the zeroth approximation we take

$$\theta_0 = \xi - \Delta, \tag{1.10}$$

$$\tau_0 = \frac{\Delta}{1-\beta}, \tag{1.11}$$

which corresponds to ice with an infinitesimal specific heat. Inserting (1.10) into the right-hand side of Eqs. (1.8) and (1.9), we obtain

$$\theta_1 = \xi - \Delta + \frac{\alpha(1-\beta)(\Delta^2 - \xi^2)}{2(1+\alpha\Delta)}, \tag{1.12}$$

$$\tau_1 = \frac{\Delta}{1-\beta}\left(1 + \frac{\alpha\Delta}{2}\right). \tag{1.13}$$

Continuing the iteration process, we will obtain after the second approximation:

$$\theta_2 = \xi - \Delta + \alpha(1-\beta)\frac{A_4\xi^4 + A_3\xi^2 + A_2\xi + A_1}{1 + \alpha A_2}, \tag{1.14}$$

$$\tau_2 = \frac{\Delta}{1-\beta}\left(1 + \frac{\alpha\Delta}{2}\right) - \frac{\alpha^2\Delta^3}{3(1+\alpha\Delta)}, \tag{1.15}$$

where

$$A_1 = \frac{\Delta^2}{2}\left[-1 + \alpha(1-\beta)\frac{\Delta\left(1 + \frac{3}{4}\alpha\Delta\right)}{(1+\alpha\Delta)^2}\right],$$

$$A_2 = \Delta - \alpha(1-\beta)\frac{\Delta(2+\alpha\Delta)}{4(1+\alpha\Delta)},$$

$$A_3 = -\frac{1}{2} + \alpha(1-\beta)\frac{\Delta(2+\alpha\Delta)}{4(1+\alpha\Delta)^2}, \quad A_4 = \frac{\alpha^2(1-\beta)}{24(1+\alpha\Delta)^2}.$$

Since the expression for θ_3 is rather unwieldy, we will show only the expression for τ_3:

$$\tau_3 = \frac{\Delta}{1-\beta} - \frac{\alpha}{1-\beta}\int\limits_{0}^{\Delta}\left\{-x + \alpha(1-\beta)\left[\frac{A_4'\frac{x^5}{5} + A_3'\frac{x^3}{3} + A_2'x^2 + A_1'x}{1 + \alpha A_2} - \frac{\alpha A_2'\left(A_4\frac{x^5}{5} + A_3\frac{x^3}{3} + A_2x^2 + A_1x\right)}{(1+\alpha A_2)^2}\right]\right\}dx, \tag{1.16}$$

where $A' = dA/dx$.

The convergence of the iteration process is easily established by an analysis of the results in Table 1, where the numerical values for the first, the second, and the third approximation are given. The numerical solutions to the problem for several specific values of the governing parameters are given in Table 2.

TABLE 2. Freezing Time for an Ice Layer of Thickness Δ (q_W = const)

α	β	Δ								
		0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
0,05	0,01	0,1012	0,2029	0,3052	0,4079	0,5112	0,6149	0,7191	0,8239	0,9291
	0,1	0,1113	0,2232	0,3357	0,4487	0,5623	0,6764	0,7911	0,9063	1,0220
	0,2	0,1252	0,2511	0,3777	0,5048	0,6326	0,7610	0,8900	1,0196	1,1498
	0,3	0,1431	0,2870	0,4316	0,5769	0,7230	0,8697	1,0171	1,1653	1,3141
	0,4	0,1670	0,3349	0,5036	0,6731	0,8435	1,0147	1,1867	1,3595	1,4462
	0,5	0,2004	0,4018	0,6043	0,8077	1,0122	1,2176	1,4240	1,6314	1,8398
	0,6	0,2499	0,5023	0,7554	1,0099	1,2652	1,5220	1,7801	2,0393	2,2998
0,2	0,01	0,1018	0,2058	0,3117	0,4196	0,5294	0,6412	0,7548	0,8704	0,9875
	0,1	0,1120	0,2263	0,3429	0,4616	0,5824	0,7054	0,8304	0,9545	1,0865
	0,2	0,1261	0,2547	0,3857	0,5193	0,6553	0,7936	0,9343	1,0773	1,2226
	0,3	0,1441	0,2911	0,4409	0,5935	0,7489	0,9071	1,0680	1,2315	1,3976
	0,4	0,1681	0,3396	0,5144	0,6925	0,8738	1,0584	1,2461	1,4370	1,6309
	0,5	0,2017	0,4075	0,6173	0,8310	1,0487	1,2702	1,4956	1,7247	1,9576
	0,6	0,2522	0,5094	0,7716	1,0388	1,3109	1,5879	1,8697	2,1563	2,4476
0,6	0,01	0,1019	0,2103	0,3251	0,4459	0,5726	0,7048	0,8422	0,9844	1,1312
	0,1	0,1125	0,2321	0,3585	0,4916	0,6311	0,7767	0,9281	1,0849	1,2470
	0,2	0,1270	0,2618	0,4043	0,5543	0,7113	0,8754	1,0460	1,2230	1,4059
	0,3	0,1455	0,3000	0,4630	0,6346	0,8143	1,0020	1,1974	1,4002	1,6101
	0,4	0,1702	0,3507	0,5411	0,7415	0,9514	1,1707	1,3991	1,6363	1,8822
	0,5	0,2046	0,4215	0,6503	0,8909	1,1431	1,4066	1,6812	1,9667	2,2627
	0,6	0,2562	0,5275	0,8137	1,1147	1,4303	1,7601	2,1041	2,4619	2,8334

2. Freezing of a Liquid at a Convectively Cooled Flat Wall

Beginning at time t = 0, let the infinitely large plate x = 0 be sprayed with a coolant the temperature of which, $T_0 < T_f$, is constant. The coefficient of heat transfer h_W between the cooled wall and the heat carrier will also be assumed constant, and the thermal resistance of the plate will be considered negligibly low. Function F[t, T(0, t)] then becomes

$$F[t,\ T(0,\ t)] = h_w[T(0,\ t) - T_0]. \tag{2.1}$$

The procedure for solving problem (1)-(5) with condition (2.1) is in many aspects analogous to the procedure followed in the preceding case and, therefore, we will omit here the intermediate steps and will show the respective final integrodifferential equation of the temperature distribution in a frozen layer as well as the expression for the time after which an ice layer has attained a specified thickness:

$$\theta(\xi,\ \Delta) = \gamma\left[1 + \frac{\alpha-\gamma}{\gamma}\xi\right] + (\alpha-\gamma)(1-\Delta)\,\frac{1 + \dfrac{\alpha-\gamma}{\gamma}\xi - \varphi\left[\displaystyle\int_0^\Delta \frac{\partial\theta}{\partial\Delta}d\xi + \frac{\alpha-\gamma}{\gamma}\int_0^\xi\int_\xi^\Delta \frac{\partial\theta}{\partial\Delta}d\xi d\xi\right]}{1 + \dfrac{\alpha-\gamma}{\gamma}\Delta - \varphi\left[\displaystyle\int_0^\Delta \frac{\partial\theta}{\partial\Delta}d\xi + \frac{\alpha-\gamma}{\gamma}\int_0^\Delta\int_\xi^\Delta \frac{\partial\theta}{\partial\Delta}d\xi d\xi\right]}, \tag{2.2}$$

$$\tau = \frac{1}{\gamma\varphi}\int_0^\Delta \frac{1 + \dfrac{\alpha-\gamma}{\gamma}\Delta - \varphi\left[\displaystyle\int_0^\Delta \frac{d\theta}{\partial\Delta}d\xi + \frac{\alpha-\gamma}{\gamma}\int_0^\Delta\int_\xi^\Delta \frac{\partial\theta}{\partial\Delta}d\xi d\xi\right]}{1-\Delta}d\Delta, \tag{2.3}$$

where

$$\theta = \frac{T-T_0}{T_e-T_f};\quad \tau = \frac{h_w^2}{\rho c k}t;\quad \xi = \frac{x}{\delta_s};$$

$$\delta_s = \frac{k}{h_w}\left(\frac{h_w}{h_e}\cdot\frac{T_f-T_0}{T_e-T_f} - 1\right);\quad \Delta = \frac{\delta}{\delta_s};$$

$$\alpha = \frac{T_f-T_0}{T_e-T_f};\quad \gamma = \frac{h_e}{h_w};\quad \varphi = \frac{c(T_e-T_f)}{L}.$$

In order to obtain the zeroth approximation, we set $\partial\theta/\partial\Delta = 0$, corresponding to ice with almost zero specific heat. We then have

TABLE 3. Freezing Time for an Ice Layer of Thickness Δ at a Convectively Cooled Wall

γ, φ	α	Δ = 0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
	10	2,7676	9,5109	21,2267	39;5394	66,9805	107,778	169,807	270,541	464,269
γ=0,5	15	4,7406	16,3135	36,8348	69,4214	118,9403	193,587	308,836	499,839	881,776
φ=0,2	20	6,4713	23,2549	53,6812	102,6488	177,9317	292,376	470,436	767,586	1365,614
	10	1,7092	6,0883	13,8332	26,1096	44,7462	72,7968	115,951	186,833	324,673
γ=0,5	15	3,1571	11,1971	25,7071	49,1059	85,1771	140,294	226,493	371,113	663,737
φ=0,4	20	4,5503	16,7947	39,3810	76,3240	183,8890	222,626	362,454	359,861	1079,37
	10	0,9794	3,0068	6,3063	11,2744	18,5395	29,1561	45,0908	70,7091	119,577
γ=1,0	15	1,5454	4,8517	10,4343	19,0611	31,9441	51,1300	80,4855	128,796	224,852
φ=0,2	20	2,0196	6,6958	14,8326	27,6490	47,0615	76,3020	121,474	153,742	346,661
	10	0,5852	1,8883	4,0650	7,3980	12,3395	19,6503	30,7505	48,793	83,579
γ=1,0	15	0,9991	3,2858	7,2357	13,4409	22,8423	37,0307	59,0129	95,621	169,245
φ=0,4	20	1,3836	4,7883	10,8359	20,5156	35,3886	58,0897	93,5968	153,219	274,017
	10	0,5551	1,5943	3,2118	5,5802	8,9765	13,8678	21,1261	32,6887	54,5771
γ=1,5	15	0,8342	2,4712	5,1402	9,1766	15,1161	23,8664	37,1443	58,8523	101,779
φ=0,2	20	1,0599	3,3343	7,1737	13,1125	21,9984	35,2661	55,6269	89,2423	156,303
	10	0,3240	0,9831	2,0437	3,6288	5,9373	9,3055	14,3626	22,5068	38,0849
γ=1,5	15	0,5264	1,6494	3,5340	6,4378	10,7758	17,2530	27 2037	43,6621	76,5720
φ=0,4	20	0,7093	2,3567	5,2091	9,6079	16,5131	26,8237	42,8403	69,5872	123,529

$$\theta_0 = \gamma\left[1 + \frac{\alpha-\gamma}{\gamma}\xi\right] + (\alpha-\gamma)(1-\Delta)\frac{1+\frac{\alpha-\gamma}{\gamma}\xi}{1+\frac{\alpha-\gamma}{\gamma}\Delta}, \tag{2.4}$$

$$\tau_0 = -\frac{\alpha}{\varphi\gamma^2}\left[\ln(1-\Delta) + \frac{\alpha-\gamma}{\alpha}\Delta\right]. \tag{2.5}$$

The first approximation yields:

$$\theta_1 = \gamma + (\alpha-\gamma)\xi + (\alpha-\gamma)(1-\Delta)\frac{1+\frac{\alpha-\gamma}{\gamma}\xi - \gamma\Phi(\Delta,\xi)}{1+\frac{\alpha-\gamma}{\gamma}\Delta - \varphi\Phi(\Delta,\Delta)}, \tag{2.6}$$

$$\tau_1 = \frac{1}{\gamma\varphi}\int_0^{\Delta}\frac{\left\{1+\frac{\alpha-\gamma}{\gamma}\Delta - \varphi\left[A\Delta\left(1+\frac{\alpha-\gamma}{2\gamma}\Delta\right) + \frac{\alpha-\gamma}{2}A\Delta^2\left(\frac{1}{2} - \frac{\alpha-\gamma}{3\gamma}\Delta\right)\right]\right\}d\Delta}{1-\Delta}, \tag{2.7}$$

where

$$A = -\alpha\frac{\alpha-\gamma}{\gamma}\left(1 - \frac{\alpha-\gamma}{\gamma}\Delta\right)^{-2},$$

$$\Phi(\Delta,\xi) = A\left\{\Delta + \frac{\alpha-\gamma}{2\gamma}\Delta^2 + \frac{\alpha-\gamma}{\gamma}\left(\Delta + \frac{\alpha-\gamma}{2\gamma}\Delta^2\right)\xi - \frac{\alpha-\gamma}{2\gamma}\xi^2 - \left(\frac{\alpha-\gamma}{\gamma}\right)^2\frac{\xi^3}{6}\right\}.$$

Since the expression for θ_2 is very unwieldy, we will show only the expression for τ_2:

$$\tau_2 = \frac{1}{\gamma\varphi}\int_0^{\Delta}\frac{1+\frac{\alpha-\gamma}{\gamma}\Delta - \varphi\left(J_1 + \frac{\alpha-\gamma}{\gamma}J_2\right)}{1-\Delta}d\Delta, \tag{2.8}$$

where

$$J_1 = \frac{\gamma-\alpha}{q} + \frac{(\alpha-\gamma)(1-\Delta)}{q^2}\left(\varphi B - \frac{\alpha-\gamma}{\gamma}\right)$$

$$\times\left(\Delta + \frac{\alpha-\gamma}{2\gamma}\Delta^2 - \varphi A C_1\right) - \frac{\varphi(\alpha-\gamma)(1-\Delta)}{q}$$

$$\times\left[A'C_1 + A\left(\Delta + \frac{\alpha-\gamma}{\gamma}\Delta^2\right) + \frac{\alpha-\gamma}{\gamma}\left(1 + \frac{\alpha-\gamma}{\gamma}\Delta\right)\frac{\Delta^2}{2}\right];$$

$$M = \frac{\Delta^3}{2} + \frac{\alpha-\gamma}{\gamma}\left(\frac{11}{24} + \frac{\alpha-\gamma}{6\gamma}\Delta\right)\Delta^4 - \left(\frac{\alpha-\gamma}{\gamma}\right)^2\frac{\Delta^5}{30};$$

$$R = \frac{\gamma-\alpha}{q} + \frac{(\alpha-\gamma)(1-\Delta)}{q^2}\left(\varphi B - \frac{\alpha-\gamma}{\gamma}\right);$$

$$Q = -\frac{(\alpha-\gamma)(1-\Delta)\varphi}{q};\quad q = 1 + \frac{\alpha-\gamma}{\gamma}\Delta - \varphi\Phi(\Delta,\Delta);$$

$$J_2 = R\left(\frac{\Delta^2}{2} + \frac{\alpha-\gamma}{3\gamma}\Delta^3 - \varphi AM\right) + Q\left\{A'M + A\left[\frac{\Delta^2}{2} + \frac{\alpha-\gamma}{\gamma}\left(\frac{5}{6} + \frac{\alpha-\gamma}{3\gamma}\Delta\right)\Delta^3\right]\right\};$$

$$B = (A'\Delta + A)\left\{1 + \frac{1}{2}\frac{\alpha-\gamma}{\gamma}\Delta + \frac{\alpha-\gamma}{\gamma}\left[\frac{\Delta}{2}\left(1 + \frac{\alpha-\gamma}{\gamma}\Delta\right) - \frac{\alpha-\gamma}{6\gamma}\Delta^2\right]\right\} + A\Delta\frac{\alpha-\gamma}{\gamma}\left[1 + \frac{2}{3}\frac{\alpha-\gamma}{\gamma}\Delta\right];$$

$$C_1 = \Delta^2 + \frac{\alpha-\gamma}{2\gamma}\Delta^3 + \frac{\alpha-\gamma}{\gamma}\left(\Delta + \frac{\alpha-\gamma}{2\gamma}\Delta^2\right)\frac{\Delta^2}{2} - \frac{\alpha-\gamma}{6\gamma}\Delta^3 - \left(\frac{\alpha-\gamma}{\gamma}\right)^2\frac{\Delta^4}{24};\quad A' = \frac{dA}{d\Delta}.$$

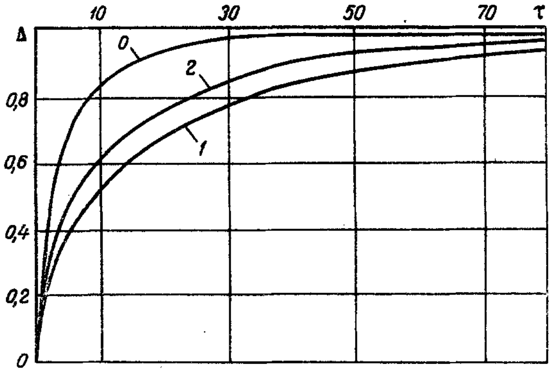

Fig. 1. Thickness of a frozen ice layer as a function of time ($\alpha = 10$, $\gamma = 1.5$, $\varphi = 0.4$): 0) $\Delta_0(\tau)$; 1) $\Delta_1(\tau)$; 2) $\Delta_2(\tau)$.

Calculated values of τ_0, τ_1, and τ_2 are shown in Fig. 1 for the case where $\alpha = 10$, $\gamma = 1.5$; $\varphi = 0.4$. It is evident here that between the second and the first approximation the maximum relative discrepancy in the results at $\tau = 15$ is half as large as that between the first and the zeroth approximation. Values of τ_2 as a function of Δ are given in Table 3 for several specific values of the governing parameters.

NOTATION

T	is the temperature distribution;
x	is the space coordinate;
t	is the time;
δ	is the thickness of the frozen layer;
T_e	is the temperature of liquid;
T_f	is the temperature of phase transition;
T_0	is the temperature of coolant;
ρ	is the density of ice;
c	is the specific heat of ice;
k	is the thermal conductivity of ice;
L	is the latent heat of fusion;
q_w	is the heat flux extracted from the wall;
h_w	is the coefficient of heat transfer between wall and coolant;
h_e	is the coefficient of heat transfer between the liquid and the phase-transition surface;
D	is the region of the phase plane analyzed in the problem;
δ_s	is the maximum thickness of the frozen layer;
z	is the characteristic linear dimension of the problem;
θ	is the dimensionless temperature distribution;
ξ	is the dimensionless space coordinate;
τ	is the dimensionless time coordinate;
Δ	is the dimensionless thickness of the frozen layer;
α, β, γ, φ	are dimensionless parameters of the problem;
θ_0, θ_1, θ_2, θ_3	are dimensionless temperature distributions;
τ_0, τ_1, τ_2, τ_3	are the dimensionless times after the respective approximation.

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